

Mathematical Journal of Okayama University

Volume 43, Issue 1

2001

Article 8

JANUARY 2001

Adjoint Orbit Types of Compact Exceptional Lie Group G_2 in its Lie Algebra

Takashi Miyasaka*

*Takatoh High School

Copyright ©2001 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

Math. J. Okayama Univ. **43**(2001), 17–23

ADJOINT ORBIT TYPES OF COMPACT EXCEPTIONAL LIE GROUP G_2 IN ITS LIE ALGEBRA

TAKASHI MIYASAKA

INTRODUCTION

A Lie group G naturally acts on its Lie algebra \mathfrak{g} , called the adjoint action. In this paper, we determine the orbit types of the compact exceptional Lie group G_2 in its Lie algebra \mathfrak{g}_2 . As results, the group G_2 has four orbit types in the Lie algebra \mathfrak{g}_2 as

$$\begin{aligned} G_2/G_2, & & G_2/(U(1) \times U(1)), \\ G_2/((Sp(1) \times U(1))/\mathbf{Z}_2), & & G_2/((U(1) \times Sp(1))/\mathbf{Z}_2). \end{aligned}$$

These orbits, especially the last two orbits, are not equivalent, that is, there exists no G_2 -equivariant homeomorphism among them. Finally, the author would like to thank Professor Ichiro Yokota for his earnest guidance, useful advice and constant encouragement.

0. PRELIMINARIES AND NOTATION

- (1) For a group G and an element s of G , \tilde{s} denotes the inner automorphism induced by s :

$$\tilde{s}(g) = sgs^{-1}, \quad g \in G,$$

then $G^{\tilde{s}} = \{g \in G \mid sg = gs\}$. Hereafter $G^{\tilde{s}}$ is briefly written by G^s .

- (2) For a transformation group G of a space M , the isotropy subgroup of G at a point $m \in M$ is denoted by G_m :

$$G_m = \{g \in G \mid gm = m\}.$$

- (3) As mentioned in the introduction, a Lie group G acts on its Lie algebra \mathfrak{g} . When G is a compact Lie group, any element $X \in \mathfrak{g}$ is transformed to some element D of a fixed Cartan subalgebra \mathfrak{h} . Hence, to determine the conjugate classes of isotropy subgroups G_X , it suffices to consider the case of $X = D \in \mathfrak{h}$.

1. THE CAYLEY ALGEBRA \mathfrak{C} AND THE GROUP G_2

Let \mathfrak{C} be the division Cayley algebra with the canonical basis $\{e_0 = 1, e_1, \dots, e_7\}$ and in \mathfrak{C} the conjugation \bar{x} , the inner product (x, y) and the length $|x|$ are naturally defined ([1], [3]). The Cayley algebra \mathfrak{C} contains

naturally the field \mathbf{R} of real numbers, furthermore the field \mathbf{C} of complex numbers and the field \mathbf{H} of quaternion numbers:

$$\begin{aligned}\mathbf{R} &= \{x = x1 \mid x \in \mathbf{R}\}, \\ \mathbf{C} &= \{x_0 + x_1e_1 \mid x_0, x_1 \in \mathbf{R}\}, \\ \mathbf{H} &= \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 \mid x_0, x_1, x_2, x_3 \in \mathbf{R}\}.\end{aligned}$$

The automorphism group G_2 of the Cayley algebra \mathfrak{C} :

$$G_2 = \{\alpha \in \text{Iso}_R(\mathfrak{C}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}$$

is the simply connected compact Lie group of type G_2 .

Any element $x \in \mathfrak{C}$ can be expressed as $x = a + be_4$, $a, b \in \mathbf{H}$, and \mathfrak{C} is isomorphic to the algebra $\mathbf{H} \oplus \mathbf{H}e_4$ with multiplication

$$(a + be_4)(c + de_4) = (ac - \bar{d}b) + (b\bar{c} + da)e_4.$$

We define an \mathbf{R} -linear transformation γ of \mathfrak{C} by

$$\gamma(a + be_4) = a - be_4, \quad a + be_4 \in \mathbf{H} \oplus \mathbf{H}e_4 = \mathfrak{C}.$$

Next, to an element

$$x = a + m_1e_2 + m_2e_4 + m_3e_6, \quad a, m_1, m_2, m_3 \in \mathbf{C}$$

of \mathfrak{C} , we associate an element

$$a + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

of the algebra $\mathbf{C} \oplus \mathbf{C}^3$ with the multiplication

$$(a + \mathbf{m})(b + \mathbf{n}) = (ab - \langle \mathbf{m}, \mathbf{n} \rangle) + (a\mathbf{n} + \bar{b}\mathbf{m} - \overline{\mathbf{m} \times \mathbf{n}}),$$

where $\langle \mathbf{m}, \mathbf{n} \rangle = {}^t\mathbf{m}\bar{\mathbf{n}}$ and $\mathbf{m} \times \mathbf{n} \in \mathbf{C}^3$ is the exterior product of \mathbf{m}, \mathbf{n} . Note that $\mathbf{C} \oplus \mathbf{C}^3$ is a left \mathbf{C} -module. Hereafter we identify \mathfrak{C} with $\mathbf{H} \oplus \mathbf{H}e_4$ and $\mathbf{C} \oplus \mathbf{C}^3$:

$$\mathfrak{C} = \mathbf{H} \oplus \mathbf{H}e_4, \quad \mathfrak{C} = \mathbf{C} \oplus \mathbf{C}^3.$$

We define an \mathbf{R} -linear transformation $\gamma_{\mathfrak{C}}$ of \mathfrak{C} by

$$\gamma_{\mathfrak{C}}(a + \mathbf{m}) = \bar{a} + \bar{\mathbf{m}}, \quad a + \mathbf{m} \in \mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C}.$$

Then, $\gamma, \gamma_{\mathfrak{C}} \in G_2$, $\gamma^2 = \gamma_{\mathfrak{C}}^2 = 1$ and $\gamma, \gamma_{\mathfrak{C}}$ are conjugate in G_2 ([1]).

2. SUBGROUPS $(Sp(1) \times Sp(1))/\mathbf{Z}_2$ AND $SU(3)$ OF G_2

Proposition 1 ([3]). $(G_2)^\gamma \cong (Sp(1) \times Sp(1))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, 1), (-1, -1)\}$.

Proof. Let $Sp(1) = \{p \in \mathbf{H} \mid p\bar{p} = 1\}$ and we define a map $\varphi : Sp(1) \times Sp(1) \rightarrow (G_2)^\gamma$ by

$$\varphi(p, q)(a + be_4) = qa\bar{q} + (pb\bar{q})e_4, \quad a + be_4 \in \mathbf{H} \oplus \mathbf{H}e_4 = \mathfrak{C}.$$

φ is well-defined: $\varphi(p, q) \in (G_2)^\gamma$ and φ is a homomorphism. We shall show that φ is onto. Let $\alpha \in (G_2)^\gamma$. Note that

$$\mathfrak{C}_\gamma = \{x \in \mathfrak{C} \mid \gamma x = x\} = \mathbf{H}, \quad \mathfrak{C}_{-\gamma} = \{x \in \mathfrak{C} \mid \gamma x = -x\} = \mathbf{H}e_4$$

and these spaces are invariant under the action of the group $(G_2)^\gamma$. Now, since $\alpha \in (G_2)^\gamma$ induces an automorphism of $\mathfrak{C}_\gamma = \mathbf{H}$, there exists $q \in Sp(1)$ such that

$$\alpha a = qa\bar{q}, \quad a \in \mathbf{H}.$$

Let $\beta = \varphi(1, q)^{-1}\alpha$. Then $\beta \in (G_2)^\gamma$ and $\beta|_{\mathbf{H}} = 1$. Since β induces an endomorphism of $\mathbf{H}e_4$, there exists $p \in \mathbf{H}$ such that $\beta e_4 = pe_4$. From $|p| = |pe_4| = |\beta e_4| = |e_4| = 1$, we see that $p \in Sp(1)$. Then

$$\beta(a + be_4) = \beta a + (\beta b)(\beta e_4) = a + b(pe_4) = a + (pb)e_4 = \varphi(p, 1)(a + be_4).$$

Hence, $\beta = \varphi(p, 1)$ and $\alpha = \varphi(1, q)\varphi(p, 1) = \varphi(p, q)$ which shows that φ is onto. $\ker \varphi = \{(1, 1), (-1, -1)\} = \mathbf{Z}_2$. Thus, we have $(G_2)^\gamma \cong (Sp(1) \times Sp(1))/\mathbf{Z}_2$. \square

Proposition 2 ([1], [3]). $(G_2)_{e_1} \cong SU(3)$.

Proof. Let $SU(3) = \{A \in M(3, \mathbf{C}) \mid A^*A = E, \det A = 1\}$ and we define a map $\psi : SU(3) \rightarrow (G_2)_{e_1}$ by

$$\psi(A)(a + \mathbf{m}) = a + A\mathbf{m}, \quad a + \mathbf{m} \in \mathbf{C} \oplus \mathbf{C}^3 = \mathfrak{C}.$$

ψ is well-defined: $\psi(A) \in (G_2)_{e_1}$. ψ is injective and a homomorphism. We shall show that ψ is onto. Let $\alpha \in (G_2)_{e_1}$. Note that α induces a \mathbf{C} -linear transformation of \mathbf{C}^3 . Now let

$$\alpha e_2 = \mathbf{a}_1, \quad \alpha e_4 = \mathbf{a}_2, \quad \alpha e_6 = \mathbf{a}_3$$

and consider a matrix $A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in M(3, \mathbf{C})$. From $(\alpha e_2)(\alpha e_4) = \alpha(e_2 e_4) = -\alpha(e_6)$, we have $\mathbf{a}_1 \mathbf{a}_2 = -\mathbf{a}_3$, namely, $-\langle \mathbf{a}_1, \mathbf{a}_2 \rangle - \overline{\mathbf{a}_1} \times \mathbf{a}_2 = -\mathbf{a}_3$, then

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = 0, \quad \mathbf{a}_3 = \overline{\mathbf{a}_1} \times \mathbf{a}_2.$$

Similarly, we have $\langle \mathbf{a}_2, \mathbf{a}_3 \rangle = \langle \mathbf{a}_3, \mathbf{a}_1 \rangle = 0$. Moreover, from $(\alpha e_k)(\alpha e_k) = \alpha(e_k e_k) = \alpha(-1) = -1$, we have $\langle \mathbf{a}_k, \mathbf{a}_k \rangle = 1$, hence, $A \in U(3) = \{A \in M(3, \mathbf{C}) \mid A^*A = E\}$. Finally, $\det A = (\mathbf{a}_3, \mathbf{a}_1 \times \mathbf{a}_2) = (\mathbf{a}_3, \overline{\mathbf{a}_1} \times \mathbf{a}_2) = \langle \mathbf{a}_3, \mathbf{a}_3 \rangle = 1$ (where (\mathbf{a}, \mathbf{b}) is the usual inner product in \mathbf{C}^3 : $(\mathbf{a}, \mathbf{b}) = {}^t \mathbf{a} \mathbf{b}$). Hence, we

have $A \in SU(3)$ and $\psi(A) = \alpha$ which shows that ψ is onto. Thus, we have $SU(3) \cong (G_2)_{e_1}$. \square

3. ADJOINT ORBIT TYPES OF G_2 IN THE LIE ALGEBRA \mathfrak{g}_2

The Lie algebra \mathfrak{g}_2 of the group G_2 is given by

$$\mathfrak{g}_2 = \{D \in \text{Hom}_R(\mathfrak{C}) \mid D(xy) = (Dx)y + x(Dy)\}.$$

The adjoint action of the group G_2 on the Lie algebra \mathfrak{g}_2 is given by

$$\mu : G_2 \times \mathfrak{g}_2 \rightarrow \mathfrak{g}_2, \quad \mu(\alpha, D) = \alpha D \alpha^{-1}.$$

Now, we shall determine the adjoint orbit types of the group G_2 in \mathfrak{g}_2 . Since the group G_2 contains the subgroup $SU(3)$ (Proposition 2), the Lie algebra \mathfrak{g}_2 also contains the subalgebra $\mathfrak{su}(3) = \{D \in M(3, \mathbf{C}) \mid D^* + D = 0, \text{tr}(D) = 0\}$. We choose a Cartan subalgebra \mathfrak{h} of $\mathfrak{su}(3)$ as

$$\mathfrak{h} = \left\{ D(r, s, t) = \begin{pmatrix} re_1 & 0 & 0 \\ 0 & se_1 & 0 \\ 0 & 0 & te_1 \end{pmatrix} \mid r, s, t \in \mathbf{R}, r + s + t = 0 \right\},$$

which is also a Cartan subalgebra of \mathfrak{g}_2 . The order of r, s, t of $D(r, s, t)$ is arbitrarily exchanged by the action of G_2 .

Note that between the induced mappings $\varphi_* : \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \rightarrow \mathfrak{g}_2$ and $\psi_* : \mathfrak{su}(3) \rightarrow \mathfrak{g}_2$ of φ and ψ , there exist the following relations:

$$\varphi_*(e_1, 0) = \psi_*(\text{diag}(0, e_1, -e_1)), \quad \varphi_*(0, e_1) = \psi_*(\text{diag}(2e_1, -e_1, -e_1)).$$

Theorem 3. *The orbit types in \mathfrak{g}_2 through $D(r, s, t) \in \mathfrak{h}$ under the adjoint action of the group G_2 are as follows:*

- (1) *In the case: $r = s = t = 0$, the orbit through $D(0, 0, 0)$ is*

$$G_2/G_2.$$

- (2) *In the case: r, s, t are non-zero and distinct, the orbit through $D(r, s, t)$ is*

$$G_2/(U(1) \times U(1)).$$

- (3) *In the case: r is non-zero, the orbit through $D(2r, -r, -r)$ is*

$$G_2/((Sp(1) \times U(1))/\mathbf{Z}_2).$$

- (4) *In the case: r is non-zero, the orbit through $D(0, r, -r)$ is*

$$G_2/((U(1) \times Sp(1))/\mathbf{Z}_2).$$

Proof. (1) trivial.

(2) Let $U(1) = \{a \in \mathbf{C} \mid a\bar{a} = 1\}$ and $S(U(1) \times U(1) \times U(1))$ be the diagonal subgroup of $SU(3)$ and $\psi : S(U(1) \times U(1) \times U(1)) \rightarrow (G_2)_{D(r,s,t)}$ be the restriction map ψ of Proposition 2. Then, ψ is well-defined and

injective. We shall show that ψ is onto. For this purpose, we first show that for $\alpha \in (G_2)_{D(r,s,t)}$, we have

$$\alpha e_1 = \pm e_1.$$

Indeed, let $\alpha e_1 = a + \mathbf{m} \in \mathbf{C} \oplus \mathbf{C}^3$. From the condition $\alpha D(r, s, t) = D(r, s, t)\alpha$,

$$0 = \alpha 0 = \alpha D(r, s, t)e_1 = D(r, s, t)\alpha e_1 = D(r, s, t)(a + \mathbf{m}) = D(r, s, t)\mathbf{m},$$

we have $\mathbf{m} = 0$. So $\alpha e_1 = a$. Since $|a| = |\alpha e_1| = |e_1| = 1$ and $\alpha 1 = 1$, we have $\alpha e_1 = \pm e_1$. In the case of $\alpha e_1 = -e_1$, consider $\gamma_c \in G_2$. Then $\alpha \gamma_c e_1 = e_1$, so $\alpha \gamma_c = \psi(A)$, $A = (a_{kl}) \in SU(3)$ (Proposition 2). Hence, $\alpha = \psi(A)\gamma_c$. Then

$$A\gamma_c D(r, s, t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A\gamma_c \begin{pmatrix} re_1 \\ 0 \\ 0 \end{pmatrix} = A \begin{pmatrix} -re_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -ra_{11}e_1 \\ -ra_{21}e_1 \\ -ra_{31}e_1 \end{pmatrix}.$$

On the other hand,

$$D(r, s, t)A\gamma_c \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = D(r, s, t)A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = D(r, s, t) \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} re_1 a_{11} \\ se_1 a_{21} \\ te_1 a_{31} \end{pmatrix}.$$

From the condition $\alpha D(r, s, t) = D(r, s, t)\alpha$, we have

$${}^t(-ra_{11}e_1, -ra_{21}e_1, -ra_{31}e_1) = {}^t(re_1 a_{11}, se_1 a_{21}, te_1 a_{31}).$$

But, it is easy to see that this is false. Hence, we have $\alpha e_1 = e_1$, so $\alpha = \psi(A)$, $A \in SU(3)$. From the condition $\alpha D(r, s, t) = D(r, s, t)\alpha$ again, we have $\alpha = \psi(A)$, $A \in S(U(1) \times U(1) \times U(1))$, which shows that ψ is onto. Thus we have $(G_2)_{D(r,s,t)} = S(U(1) \times U(1) \times U(1)) \cong U(1) \times U(1)$.

(3) Since

$$\begin{aligned} \exp(\pi D(2, -1, -1)) &= \exp(\pi \psi_*(\text{diag}(2e_1, -e_1, -e_1))) \\ &= \exp(\pi \varphi_*(0, e_1)) = \gamma, \end{aligned}$$

if $\alpha \in G_2$ commutes with $D(2, -1, -1)$, then α also commutes with γ . Hence, $\alpha \in (G_2)^\gamma = \varphi(Sp(1) \times Sp(1))$ (Proposition 1). So there exist $p, q \in Sp(1)$ such that $\alpha = \varphi(p, q)$. Again from the commutativity with $\exp\left(\frac{\pi}{2}D(2, -1, -1)\right) = \exp\left(\frac{\pi}{2}\psi_*(\text{diag}(2e_1, -e_1, -e_1))\right) = \varphi(1, e_1)$, we have $\varphi(p, q)\varphi(1, e_1) = \varphi(1, e_1)\varphi(p, q)$, hence, $\varphi(p, qe_1) = \varphi(p, e_1q)$. So $qe_1 = e_1q$, therefore $q \in \mathbf{C} \cap Sp(1) = U(1)$. Conversely, $\alpha = \varphi(p, q)$ ($p \in Sp(1)$, $q \in U(1)$) commutes with $\varphi(1, e^{te_1})$, $t \in \mathbf{R}$, so α also commutes with $\varphi_*(0, e_1) = D(2, -1, -1)$. Thus, we have $(G_2)_{D(2,-1,-1)} = \varphi(Sp(1) \times U(1)) = (Sp(1) \times U(1))/\mathbf{Z}_2$.

(4)

$$\begin{aligned}\exp(\pi D(0, 1, -1)) &= \exp(\pi \psi_*(\text{diag}(0, e_1, -e_1))) \\ &= \exp(\pi(\varphi_*(e_1, 0))) = \gamma.\end{aligned}$$

Hence, $(G_2)_{D(0,r,-r)} = (U(1) \times Sp(1))/\mathbf{Z}_2$ is proved in a similar way to (3) above. \square

Proposition 4. *If r and s are non-zero, then the groups $(G_2)_{D(0,r,-r)} \cong (U(1) \times Sp(1))/\mathbf{Z}_2$ and $(G_2)_{D(2s,-s,-s)} \cong (Sp(1) \times U(1))/\mathbf{Z}_2$ are not conjugate in the group G_2 .*

Proof. We shall prove that $D(0, r, -r)$ and $D(2s, -s, -s)$ are not conjugate under the action of the group G_2 . Suppose that there exists $\alpha \in G_2$ such that

$$\alpha(D(0, r, -r)) = (D(2s, -s, -s))\alpha.$$

Let $\alpha e_2 = a_0 + \mathbf{m} \in \mathbf{C} \oplus \mathbf{C}^3$, $\mathbf{m} = {}^t(m_1, m_2, m_3)$. Then, from

$$\alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & re_1 & 0 \\ 0 & 0 & -re_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2se_1 & 0 & 0 \\ 0 & -se_1 & 0 \\ 0 & 0 & -se_1 \end{pmatrix} \left(a_0 + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \right),$$

we have $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2sm_1e_1 \\ -sm_2e_1 \\ -sm_3e_1 \end{pmatrix}$, hence, $m_1 = m_2 = m_3 = 0$, so

$$\alpha e_2 = a_0 \in \mathbf{C}.$$

From $(\alpha e_2)(\alpha e_2) = \alpha(e_2 e_2) = \alpha(-1) = -1$, we have $a_0 a_0 = -1$, hence, $\alpha e_2 = a_0 = \pm e_1$. Similarly, we have $\alpha e_3 = \pm e_1$. Then $\alpha e_1 = (\alpha e_2)(\alpha e_3) = (\pm e_1)(\pm e_1) = \pm 1$, which is a contradiction. \square

Theorem 5. *If r and s are non-zero, then the orbit spaces*

$X = \{\alpha(D(0, r, -r))\alpha^{-1} \mid \alpha \in G_2\}$ and $Y = \{\alpha(D(2s, -s, -s))\alpha^{-1} \mid \alpha \in G_2\}$ *are not equivalent.*

Proof. Suppose that there exists a G_2 -equivariant homeomorphism $h : X \rightarrow Y$. Then, there exists $\alpha \in G_2$ such that

$$(1) \quad h(D(0, r, -r)) = \alpha(D(2s, -s, -s))\alpha^{-1}.$$

For any $\delta \in (G_2)_{D(0,r,-r)}$, we have

$$\delta(h(D(0, r, -r)))\delta^{-1} = \delta(\alpha(D(2s, -s, -s))\alpha^{-1})\delta^{-1}.$$

Since h is a G_2 -equivariant homeomorphism, we have $\tilde{\delta}h = h\tilde{\delta}$. Hence, we see

$$(2) \quad h(D(0, r, -r)) = \delta(\alpha(D(2s, -s, -s))\alpha^{-1})\delta^{-1}.$$

From (1) and (2), we have

$$\delta(\alpha(D(2s, -s, -s))\alpha^{-1})\delta^{-1} = \alpha(D(2s, -s, -s))\alpha^{-1},$$

that is, $\alpha^{-1}(\delta(\alpha(D(2s, -s, -s))\alpha^{-1})\delta^{-1})\alpha = D(2s, -s, -s)$. This implies $\alpha^{-1}\delta\alpha \in (G_2)_{D(2s, -s, -s)}$. Thus, we have

$$\alpha^{-1}((G_2)_{D(0, r, -r)})\alpha \subset (G_2)_{D(2s, -s, -s)}.$$

Since $\dim((G_2)_{D(0, r, -r)}) = \dim((G_2)_{D(2s, -s, -s)})$ (Theorem 3 (3) and (4)), the inclusion above must be equal, that is,

$$\alpha^{-1}((G_2)_{D(0, r, -r)})\alpha = (G_2)_{D(2s, -s, -s)}.$$

This means that the groups $(G_2)_{D(0, r, -r)}$ and $(G_2)_{D(2s, -s, -s)}$ are conjugate in G_2 , which contradicts Proposition 4. \square

REFERENCES

- [1] I. YOKOTA, *Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups, Part I, $G = G_2, F_4$ and E_6* , Tsukuba J. Math. **14**(1990), 185–223.
- [2] I. YOKOTA, *Simple Lie groups of classical type* (in Japanese), Gendai-Sugakusya, 1990.
- [3] I. YOKOTA, *Simple Lie groups of exceptional type* (in Japanese), Gendai-Sugakusya, 1992.

TAKASHI MIYASAKA
TAKATOH HIGH SCHOOL
OBARA, TAKATOH 396-0293, JAPAN
e-mail address: coolkai@mac.com

(Received July 13, 2001)